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# An extremal property of the normal distribution, with a discrete analog

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## Abstract

We prove, using the Brascamp-Lieb inequality, that the Gaussian measure is the only strong log-concave measure having a strong log-concavity parameter equal to its covariance matrix. We also give a similar characterization of the Poisson measure in the discrete case, using “Chebyshev’s other inequality”. We briefly discuss how these results relate to Stein and Stein–Chen methods for Gaussian and Poisson approximation, and to the Bakry–Émery calculus.

## 1 Introduction and definitions

In this paper we consider probability densities on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  which are strongly log-concave. Basic properties of log-concave and strong log-concave densities are given in the survey [SW14].

**Definition 1** *Let  $f : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \rightarrow \mathbb{R}_+$  be a density function that is not supported on any subspace of dimension  $d - 1$ . We consider the potential function  $\varphi : \mathbb{R}^d \rightarrow ]-\infty, +\infty]$  defined by  $\varphi = -\log(f)$ . The density  $f$  is said to be:*

1. *Log-concave if  $\varphi$  is convex.*
2. *Strongly log-concave if there exists a symmetric positive definite  $d \times d$  matrix  $\Sigma$  such that the ratio  $g := f/\gamma_{\mu, \Sigma}$  is log-concave where  $\gamma_{\mu, \Sigma}$  is the density of the Gaussian measure of mean  $\mu$  and covariance matrix  $\Sigma$ , denoted  $N_d(\mu, \Sigma)$ . In this case we write  $f \in SLC(\Sigma, d)$ , and refer to  $\Sigma$  as the strong log-concavity parameter of  $f$ .*

For brevity, a  $d$ -dimensional random vector  $X$  is said to belong to  $SLC(\Sigma, d)$  if it admits a density  $f$  with respect to the Lebesgue measure on  $\mathbb{R}^d$  such that  $f \in SLC(\Sigma, d)$ . Observe that in Definition 1.2, the choice of  $\mu$  is irrelevant; if the ratio  $f/\gamma_{\mu, \Sigma}$  is log-concave for some  $\mu$ , it is log-concave for all  $\mu$  (since the second derivative of  $\log \gamma_{\mu, \Sigma}$  does not depend on  $\mu$ ). For simplicity authors often choose to take  $\mu$  to be zero, or to equal the expectation of  $f$ .

We state the following two results without proof:

**Proposition 2** *A strongly log-concave measure is log-concave and when its potential is twice differentiable, belonging to  $SLC(\Sigma, d)$  is equivalent to having*

$$\varphi''(x) \succeq \Sigma^{-1} \quad \text{for any } x \in \mathbb{R}^d, \quad (1)$$

where  $\varphi''$  is the Hessian matrix of  $\varphi$  and the order relation is the natural (Loewner) partial order for semi-definite symmetric matrices. Using the fact that this order is reversed on taking inverses (see [B09, Proposition 8.6.6]), we can also write this in the form

$$(\varphi''(x))^{-1} \preceq \Sigma \quad \text{for any } x \in \mathbb{R}^d, \quad (2)$$

In the one-dimensional case, for  $\alpha > 0$ , we write  $SLC(\alpha) = SLC(\alpha, 1)$  and we have:

**Proposition 3** *A differentiable density function  $f$  is in  $SLC(\alpha)$  if and only if the function  $\frac{f'(x)}{f(x)} + \frac{x}{\alpha}$  is non-increasing in  $x$ .*

Clearly, by definition any Gaussian  $X$  belongs to  $SLC(\Sigma, d)$  with strong log-concavity parameter  $\Sigma$  equal to the covariance matrix  $\text{Cov}(X, X)$ . Indeed in general the strong log-concavity parameter  $\Sigma$  is sometimes (erroneously) called the covariance parameter. A natural question is therefore the following: if  $X$  is a random vector belonging to  $SLC(\Sigma, d)$ , can we relate the strong log-concavity parameter  $\Sigma$  to the covariance matrix  $\text{Cov}(X, X)$ ?

In this note, we answer this question by proving the inequality  $\text{Cov}(X, X) \preceq \Sigma$  (in Theorem 4). Moreover, we deduce a characterization of the Gaussian; there is equality  $\Sigma = \text{Cov}(X, X)$  if and only if  $X$  has Gaussian distribution with covariance  $\Sigma$ . In Section 2, we prove this fact using the Brascamp–Lieb inequality. In Section 3, we prove a more general characterization of Gaussian distributions, but in the restricted framework of one-dimensional distributions. In Section 4, we use similar methods to prove a characterization of Poisson distributions.

## 2 The continuous case via the Brascamp-Lieb inequality

**Theorem 4** *Suppose that the random vector  $X \in SLC(\Sigma, d)$  for some symmetric positive definite matrix  $\Sigma$ .*

(a) *Then*

$$\text{Cov}(X, X) \preceq \Sigma. \quad (3)$$

(b) *If  $\text{Cov}(X, X) = \Sigma$  then  $X$  has a multivariate normal distribution with covariance matrix  $\Sigma$ , that is  $X \sim \mathcal{N}_d(\mu, \Sigma)$  for some  $\mu$ .*

Let us recall the celebrated Brascamp-Lieb inequality [BL76], that can be thought of as a weighted Poincaré inequality, and which will be instrumental in our proof. If density  $f$  is strictly log-concave, its potential  $\varphi$  is twice continuously differentiable and  $g \in L_2(f)$  is continuously differentiable, then for  $X \sim f$

$$\text{Var}(g(X)) \leq \mathbb{E} [\nabla g(X)^T (\varphi''(X))^{-1} \nabla g(X)] . \quad (4)$$

Theorem 4 also builds upon the work of Chen and Lou [CL87] on characterization of the Gaussian distribution by the Poincaré inequality. Indeed, Corollary 2.1 in [CL87] can be stated as follows. Let  $X = (X_1, \dots, X_d)$  be a random vector such that  $\text{Var}(X_j) = \sigma_j^2 > 0$  for any  $j \in \{1, \dots, d\}$ . Define

$$U(X, \Sigma) = \sup_{g \in \mathcal{H}_X} \frac{\text{Var}(g(X))}{\mathbb{E} [\nabla^T g(X) \Sigma \nabla g(X)]} , \quad (5)$$

where  $\Sigma$  is a  $d \times d$  positive semidefinite matrix with  $\sigma_1^2, \dots, \sigma_d^2$  as its diagonal elements and  $\mathcal{H}_X = \{g \in C^1(\mathbb{R}^d) \cap L_2(X) : \mathbb{E} [\nabla^T g(X) \Sigma \nabla g(X)] > 0\}$ . Clearly taking  $g(x) = x_i$  for any  $i$ , we can deduce that  $U(X, \Sigma) \geq 1$ . However [CL87, Corollary 2.1] shows that this is sharp, by proving that  $U(X, \Sigma) = 1$  if and only if  $X$  has a multivariate normal distribution with covariance matrix  $\Sigma$ .

**Proof of Theorem 4.** Assume first that the potential  $\varphi = -\log f$  of the density  $f$  of  $X$  is twice continuously differentiable. Then combining the Brascamp-Lieb inequality (4) with the assumption (2), for any continuously differentiable  $g \in L_2(f)$  we have:

$$\begin{aligned} \text{Var}(g(X)) &\leq \mathbb{E} [\nabla g(X)^T (\varphi''(X))^{-1} \nabla g(X)] \\ &\leq \mathbb{E} [\nabla^T g(X) \Sigma \nabla g(X)] . \end{aligned} \quad (6)$$

We can deduce that Equation (3) holds; for any vector  $u \in \mathbb{R}^d$  we can take the linear function  $g(x) = \sum_{i=1}^d u_i x_i$  in (6) to deduce that  $u^T \text{Cov}(X, X) u \leq u^T \Sigma u$ . Since this holds for any  $u$ , we deduce that  $\text{Cov}(X, X) \preceq \Sigma$  in the partial order sense as claimed in part (a) of the theorem.

In general, note that approximation by convolution with Gaussian vectors allows us to reduce to the case where  $\varphi$  is twice continuously differentiable. In particular, it regularizes the potential of any strongly log-concave measure, while preserving strong-log-concavity (see [SW14], Proposition 5.5), meaning that (a) holds for all SLC  $f$ .

To deduce the case of equality stated in (b), we can restate (6) to say that if  $X \in \text{SLC}(\Sigma, d)$  then the quantity defined in (5) satisfies  $U(X, \Sigma) \leq 1$ . But we already have  $U(X, \Sigma) \geq 1$ . Hence  $U(X, \Sigma) = 1$ , which implies by [CL87, Corollary 2.1], that  $X$  is a multivariate normal distribution with covariance matrix  $\Sigma$ . ■

Notice that a careful reading of the proof of Theorem 4 shows that we can weaken the assumption in the case of equality. That is, following [CL87, Corollary 2.1], it is sufficient that  $X \in \text{SLC}(\Sigma, d)$  for some  $\Sigma$  with diagonal elements  $\Sigma_{jj} = \text{Var}(X_j) = \sigma_j^2 > 0$  for each  $j \in \{1, \dots, d\}$  to deduce that  $X \sim \mathcal{N}_d(\mu, \Sigma)$  for some  $\mu$ .

### 3 A one-dimensional approach using “Chebyshev’s other inequality”

In this paragraph we consider probability measures on a space  $\mathbb{X}$  which can be either the real line  $\mathbb{R}$  (with the Borel  $\sigma$ -algebra), the set of natural integers  $\mathbb{N}$  or the discrete interval  $\{0, \dots, N\}$ . In each case,  $\mathbb{X}$  is a totally ordered set, on which the following inequality holds:

**Proposition 5** *Let  $u, v : \mathbb{X} \rightarrow \mathbb{R}$  be two functions which are either both non-decreasing or both non-increasing. Let  $X$  be a  $\mathbb{X}$ -valued random variable such that  $\mathbb{E}[u(X)^2]$  and  $\mathbb{E}[v(X)^2]$  are both finite. We then have*

$$\mathbb{E}[u(X)v(X)] \geq \mathbb{E}[u(X)]\mathbb{E}[v(X)], \quad (7)$$

which can also be written as

$$\text{Cov}(u(X), v(X)) \geq 0. \quad (8)$$

If furthermore, we suppose that  $u$  is non-decreasing,  $v$  is strictly increasing and that the covariance  $\text{Cov}(u(X), v(X))$  is 0, then  $u$  is a constant function on the image of  $X$ .

Proposition 5 is known as “Chebyshev’s other inequality” (see for example Kingman [K78, Eq. (1.7)]), or as the FKG inequality, due to a generalization of equation (8) to the framework of finite distributive lattices, see [FKG71]. For the sake of completeness, we give here a short proof:

**Proof of Proposition 5.** We simply notice that:

$$2\text{Cov}(u(X), v(X)) = \mathbb{E}[(u(X_1) - u(X_2))(v(X_1) - v(X_2))], \quad (9)$$

where  $(X_1, X_2)$  are two independent copies of  $X$ . The monotonicity assumption on  $u$  and  $v$  shows that  $(u(X_1) - u(X_2))(v(X_1) - v(X_2))$  is non-negative for all  $X_1$  and  $X_2$ , which gives the inequality on the covariance.

If we have  $\text{Cov}(u(X), v(X)) = 0$  then  $(u(X_1) - u(X_2))(v(X_1) - v(X_2)) = 0$  a.s. But the assumption on  $v$  implies that  $u(X_1) = u(X_2)$  a.s. As  $X_1$  and  $X_2$  are independent, this means that  $u$  is constant on the image of  $X$ . ■

We use Proposition 5 to deduce the following result, which can be seen as a strengthening of Theorem 4 in the one-dimensional case (see Corollary 7). It thus provides a link between the Brascamp–Lieb inequality [BL76] and Chebyshev’s other inequality.

**Proposition 6** *Let  $X$  be a real-valued random variable with mean  $\mu$  and density  $f$ , where  $f$  is in the class  $\text{SLC}(\alpha)$  for some  $\alpha > 0$ . Let  $v \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  be strictly increasing. Then*

$$\alpha \mathbb{E}[v'(X)] \geq \mathbb{E}[(X - \mu)v(X)]. \quad (10)$$

Furthermore, if for one such function  $v$ , equality is attained in inequality (10), then  $X \sim \mathcal{N}(\mu, \frac{1}{\alpha})$  for some  $\mu \in \mathbb{R}$ .

**Proof.** We set  $u(x) := \frac{f'(x)}{f(x)} + \frac{x-\mu}{\alpha}$ , which has mean  $\mathbb{E}u(X) = 0$ . By the  $SLC(\alpha)$  assumption, we know that  $u : \mathbb{R} \rightarrow \mathbb{R}$  is non-increasing. By Proposition 5, we have  $\mathbb{E}[u(X)v(X)] = \text{Cov}(u(X), v(X)) \leq 0$ . But we have :

$$\begin{aligned} \mathbb{E}[u(X)v(X)] &= \int_{\mathbb{R}} \left( \frac{f'(x)}{f(x)} + \frac{x-\mu}{\alpha} \right) v(x)f(x)dx \\ &= \int_{\mathbb{R}} f'(x)v(x)dx + \frac{1}{\alpha} \int_{\mathbb{R}} (x-\mu)v(x)f(x)dx \\ &= \int_{\mathbb{R}} f(x) \left( \frac{x-\mu}{\alpha} v(x) - v'(x) \right) dx \\ &= \frac{1}{\alpha} \mathbb{E}[(X-\mu)v(X) - \alpha v'(X)], \end{aligned}$$

from which we deduce the inequality we wanted.

If equality is attained for some strictly increasing function  $v$ , we deduce from the equality case in Proposition 5 that  $u(X)$  is a constant random variable, thus that  $u$  is a constant function on the support of  $X$ . However, the  $SLC$  assumption on  $X$  means that the potential  $\varphi$  is convex, which tells us that the support of  $X$  (the values for which  $\phi$  is finite) is an interval.

On this interval, we can consider the solutions  $f$  of the ODE  $\frac{f'(x)}{f(x)} + \frac{x-\mu}{\alpha} = A$ , which satisfy  $f(x) = Be^{Ax - \frac{(x-\mu)^2}{2\alpha}}$ , for some constants  $A, B \in \mathbb{R}$ . The constraint that  $f$  is a probability density with mean  $\mu$  implies that  $A = 0$  and  $B = \sqrt{\frac{1}{2\pi\alpha}}$ , which means that  $X \sim \mathcal{N}(\mu, \alpha)$ . ■

An immediate corollary of Proposition 6, which is the one-dimensional case of Theorem 4, is obtained by considering the case where  $v(x) = x - \mu$ :

**Corollary 7** *Let  $X$  be a real-valued random variable with density  $f$ , where  $f$  is in the class  $SLC(\alpha)$  for some  $\alpha > 0$ . Then  $\text{Var}(X) \leq \alpha$ , with equality if and only if  $X$  is Gaussian.*

Indeed, writing  $M_r(X) = \mathbb{E}(X - \mu)^r$  for the centred moments of order  $r$  and taking  $v(x) = (x - \mu)^{2r-1}$ , for any  $X \in SLC(\alpha)$  we can deduce that  $M_{2r}(X) \leq \alpha(2r-1)M_{2r-2}(X)$ , and hence by induction  $M_{2r}(X) \leq (2r)!/r!(\alpha/2)^r$ , so the values obtained by the Gaussian are extremal, as we might expect.

**Remark 8** *Equation (10) can be viewed as a one-sided version of the Stein equation, used to establish a characterization of the Gaussian distribution when proving the Central Limit Theorem in Stein's Method [S71]. That is if, with  $\alpha = \text{Var}(X)$ , the equation (10) holds with equality for all  $v$  then it is well-known that  $X$  must be Gaussian. Here, Proposition 6 allows us to reach the same conclusion if equality holds for a single  $v$ , under the additional  $SLC$  assumption.*

## 4 A characterization of Poisson distributions.

The same strategy can be adapted to the discrete case, to study random variables supported on the natural numbers  $\mathbb{N}$ , with suitable definitions of derivative and of strong log-concavity:

**Definition 9**

1. The left-derivative  $\nabla u$  of a function  $u : \mathbb{N} \rightarrow \mathbb{R}$  is defined by  $\nabla u(0) := u(0)$ , and by  $\nabla u(n) := u(n) - u(n-1)$  for  $n \geq 1$ .
2. The right-derivative  $\nabla^* v$  of a function  $v : \mathbb{N} \rightarrow \mathbb{R}$  is defined by  $\nabla^* v(n) := -v(n+1) - v(n)$  for  $n \geq 0$ .

The operators  $\nabla$  and  $\nabla^*$  are dual up to a sign, in the sense that a simple application of summation by parts gives

$$\sum_{n=0}^{\infty} (\nabla u(n)) v(n) = - \sum_{n=0}^{\infty} u(n) \nabla^* v(n), \quad (11)$$

for every function  $u, v \in \mathcal{L}^2(\mathbb{N})$ .

**Definition 10** Consider probability mass function  $f : \mathbb{N} \rightarrow \mathbb{R}_+^*$  such that  $\sum_{k=0}^{\infty} f(k) = 1$  and  $\alpha > 0$ . We say that  $f \in SLC(\alpha)$  if sequence  $\left( \frac{\nabla f(k)}{f(k)} + \frac{k}{\alpha} \right)_{k \geq 0}$  is non-increasing in  $k$ .

Direct calculations show that  $f \in SLC(\alpha)$  if and only if  $f(1) \leq \alpha f(0)$  and

$$\forall n \geq 0, \quad \frac{f(n+1)}{f(n+2)} - \frac{f(n)}{f(n+1)} = \frac{f(n+1)^2 - f(n)f(n+2)}{f(n+1)f(n+2)} \geq \frac{1}{\alpha}. \quad (12)$$

We note that this condition was introduced as a special case of Assumption A in [C09], and was studied further in [J17]. In the case of Poisson random variables with mean  $\mu$  observe that the LHS of (12) is constant and equal to  $1/\mu$ , so Poisson random variables are  $SLC(\alpha)$  where strong log-concavity parameter  $\alpha = \mu$ . Again, we shall see that this property characterizes the Poisson family, using the following result.

**Proposition 11** Let  $X$  be a  $\mathbb{N}$ -valued random variable with mean  $\mu$  such that for every  $n \geq 0$ ,  $\mathbb{P}(X = n) = f(n)$ , where  $f \in SLC(\alpha)$ . For every strictly increasing  $v : \mathbb{N} \rightarrow \mathbb{R}$  the

$$\alpha \mathbb{E}[\nabla^* v(X)] \geq \mathbb{E}[(X - \mu)v(X)]. \quad (13)$$

Furthermore, if equality is attained in equation (13) for some  $v$ , then  $X$  is Poisson with mean  $\alpha$  (we write  $X \sim \mathcal{P}(\alpha)$ ).

**Proof.** We again apply Proposition 5 with the functions  $u(k) = \frac{\nabla f(k)}{f(k)} + \frac{k-\mu}{\alpha}$  and  $v(k)$ , yielding  $\mathbb{E}[u(X)v(X)] \leq 0$ . But :

$$\begin{aligned} \mathbb{E}[u(X)v(X)] &= \sum_{k=0}^{\infty} \left( \frac{\nabla f(k)}{f(k)} + \frac{k-\mu}{\alpha} \right) v(k) f(k) \\ &= \sum_{k=0}^{\infty} (\nabla f(k)) v(k) + \frac{1}{\alpha} \sum_{k=0}^{\infty} (k-\mu) v(k) f(k) \\ &= \frac{1}{\alpha} \sum_{k=0}^{\infty} f(k) ((k-\mu)v(k) - \alpha \nabla^* v(k)) \\ &= \frac{1}{\alpha} \mathbb{E}[(X - \mu)v(X) - \alpha \nabla^* v(X)]. \end{aligned}$$

If equality is attained for some strictly increasing  $v$ , we deduce that  $u$  is a constant function, i.e. that there is some  $\lambda \in \mathbb{R}$  such that :

$$\forall n \geq 0, \quad \frac{\nabla f(n)}{f(n)} + \frac{n}{\alpha} = \lambda. \quad (14)$$

But equation (14) with  $n = 0$  implies that  $\lambda = 1$ , and thus for  $n \geq 1$ , equation (14) takes the simpler form  $\frac{f(n-1)}{f(n)} = \frac{n}{\alpha}$ , from which we deduce that  $f(n) = \frac{f(0)\alpha^n}{n!}$  for every  $n \geq 0$ . The condition  $\sum_{n=0}^{\infty} f(n) = 1$  gives  $f(0) = e^{-\alpha}$ , and we recognize the Poisson distribution  $X \sim \mathcal{P}(\alpha)$ . ■

Again, by taking  $v(x) = x - \mu$  in (13), we can deduce that the  $SLC(\alpha)$  condition can only hold if  $\alpha \geq \text{Var}(X)$ , which we can view as a discrete counterpart of Theorem 4. Note that [J17, Lemma 5.3] showed that the same condition implies that  $\alpha \geq \mu$ .

A counterpart of Remark 8 holds on  $\mathbb{N}$ , referring to the Stein–Chen method in Poisson approximation [C75]. That is the Stein–Chen method is based on the fact that if (13) holds with equality for  $\alpha = \mu$  for every function  $v$ , then we can deduce that  $X$  must be Poisson. Again, we are able to reach the same conclusion under the SLC condition if equality is attained for a single function  $v$ .

**Remark 12** *One further link between discrete and continuous settings is the following. It is well-known that strong log-concave densities satisfy the so-called Bakry–Émery condition, which is a natural setting under which functional inequalities (including Poincaré and log-Sobolev) can be proved, with lower bounds on  $\varphi''$  of the form (1) guaranteeing bounds on the log-Sobolev constant – see for example [BGL14] for a review of this material. It is striking that [J17] proved similar results on  $\mathbb{N}$  with a similar role being played by the value of  $\alpha$  arising in the discrete SLC condition 12. The results of the current paper give further evidence of a natural link between these formulations.*

We briefly remark that similar arguments can be used to characterize the binomial distribution among random variables with probability mass functions  $f$  supported on discrete interval  $\{0, \dots, N\}$ . That is, if we define derivative  $\nabla_N h(0) = h(0)$  and

$$\nabla_N h(n) := \frac{N-n}{N}h(n) - \frac{N-n+1}{N}h(n-1) \quad \text{for } 1 \leq n \leq N, \quad (15)$$

and its conjugate to satisfy  $\nabla_N^* h(N) = 0$  and

$$\nabla_N^* h(n) := \frac{N-n}{N}(h(n+1) - h(n)) = \frac{N-n}{N}\nabla^* h(n) \quad \text{for } 0 \leq n \leq N-1, \quad (16)$$

we can define the set of  $SLC_N(\alpha)$  random variables to be those for which  $u(n) := \frac{\nabla_N f(n)}{f(n)} + \frac{n-\mu}{\alpha}$  is non-increasing in  $n$ . Observe that taking  $f$  to be Binomial( $N, p$ ) random variables, this property holds with equality if  $\alpha = Np = \mu$ .

Again, using the same argument based on Chebyshev we can deduce that for random variables  $X \in SLC_N(\alpha)$  and strictly increasing functions  $v$ , the expectation

$$\alpha \mathbb{E} [\nabla_N^* v(X)] \geq \mathbb{E} [(X - \mu)v(X)]. \quad (17)$$

Again taking  $v(x) = x - \mu$  we deduce that  $\text{Var}(X) \leq \alpha(1 - \mu/N)$ . Note that  $\text{Var}(X) = Np(1 - p) = \alpha(1 - \mu/N)$  if  $X$  is Binomial( $N, p$ ). Indeed as before, if equality in (17) holds for some strictly increasing  $v$ , a similar argument based on  $u(n)$  being constant allows us to deduce that  $f(n) = \binom{N}{n}q^n(1-q)^{N-n}$ , where  $q = \alpha/N$ , so  $f$  is Binomial with mean  $\alpha$ .

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